In the Name of GOD

Digital Modulations

In this session we want to talk about digital Modulation techniques and we want to analyse the main parameters of signal space & equi. vector space for these Modulation.

1) Amplitude Modulation (ASK)

Amplitue Shift keying or pass-band PAM

(Pulse Amplitude Modulation)

As you know in ASK Modulation, our digital information modulate the amplitude of carrier waveform. Means that based on the information symbols in the input of modulator a related amplitude is selected and then the carrier waveform is transmitted from transmitter to the receiver using this amplitude.

So, the signals we use for transmission using M-ary ASK modulation $\{S_m(t)\}_{m=1}^{M}$ are as follows;

Carrier waveform

$$S_m(t) = A_m \, g(t) \, \overbrace{\cos 2\pi f_c t}^{\text{Carrier waveform}} \qquad 1 \le m \le M, \qquad 0 \le t \le T$$

Amplitude of Carrier signal that carry our information

Carrier frequency

Symbol duration

g(t) is an specified signal used to pulse-shaping or spectrum-shapping of trasmitted signal proper for Comm. Channel characteristics

As you can see from the relation of $\{S_m(t)\}_{m=1}^{M}$,

all signals in this set are proportion to $g(t)\cos 2\pi f_c t$.

presentded a

Mean all signals in this set, are the linear combination of

$g(t)\cos 2\pi f_c t$. So, based on what we learn from Gram-Schmidt

algorithm, we can select $g(t)\cos 2\pi f_c t$ and normaliz it and take it as the basis function of the signal set spanned by $\{s_m(t)\}_{m=1}^{M}$. So, we have just one basis as follows,

$$f_1(t) = \frac{g(t)\cos 2\pi f_c t}{\| g(t) \cos 2\pi f_c t \|}$$

and the dimension of signal space is equal to the number of basis functions and so is equal to 1. $(N=1)$

we should calculate $\| g(t) \cos 2\pi f_c t \|$ to completely define $f_1(t)$ as the signal space basis function.

$$\| g(t) \cos 2\pi f_c (t) \|^2 = \int_0^T \left(g(t) \cos 2\pi f_c t \right)^2 dt$$

$$\Rightarrow \int_0^T g^2(t) \underbrace{\cos^2 2\pi f_c t}_{\frac{1}{2}(1+\cos 4\pi f_c t)} \, dt = \underbrace{\frac{1}{2}\int_0^T g^2(t)\,dt}_{\text{Energy of } g(t) = E_g} + \underbrace{\frac{1}{2}\int_0^T g^2(t)\cos 4\pi f_c t\, dt}_{0}$$

narrow-band signal

$$\Rightarrow \| g(t) \cos 2\pi f_c t \|^2 = \frac{1}{2} E_g \quad \Rightarrow \| g(t) \cos 2\pi f_c t \| = \sqrt{\frac{E_g}{2}}$$

$$\Longrightarrow f_1(t) = \frac{g(t)\cos 2\pi f_c t}{\| g(t)\cos 2\pi f_c t \|} = \underbrace{\sqrt{\frac{2}{E_g}}\; g(t)\cos 2\pi f_c t}_{\text{signal space basis function}}$$

So, we can represent all singals in the signal space such as $\{s_m(t)\}_{m=1}^{M}$ as the linear combination of $f_1(t)$ to get the equivalent vector space.

$$s_m(t) = A_m \, g(t) \cos 2\pi f_c t \qquad 1 \leq m \leq M, \quad 0 \leq t \leq T \qquad (1)$$

$$f_1(t) = \sqrt{\frac{2}{E_g}} \, g(t) \cos 2\pi f_c t \qquad (2)$$

$(1), (2) \Longrightarrow$

$$s_m(t) = \underbrace{\boxed{\sqrt{\frac{E_g}{2}} A_m}}_{s_m} f_1(t) \quad , \quad \underline{s}_m = s_m = \sqrt{\frac{E_g}{2}} A_m$$

Signal Constellation for M-ASK

Now, we want to analyse the signal space & vector space for M-ASK modulation technique and calculate the main parameters of this signalling set.

First parameter is signal energies $\{ S_m(t) \}_{m=1}^{M}$

$$S_m(t) = A_m\, g(t) \cos 2\pi f_c t = S_m\, f_1(t)$$

in which $S_m = \sqrt{\frac{\mathcal{E}_g}{2}}\, A_m$ and $f_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}}\, g(t) \cos 2\pi f_c t$

As we know the energy of $s_m(t)$ is defined as follows

$$\underbrace{E_m}_{\text{Energy of } s_m(t)} = \int_0^T s_m^2(t)\,dt = \int_0^T \left(A_m\, g(t) \cos 2\pi f_c t\right)^2 dt \underset{\text{Vector Space}}{=} \|\underline{s}_m\|^2 = \frac{\mathcal{E}_g}{2} A_m^2$$

$$s_m(t) = s_m f_1(t)$$

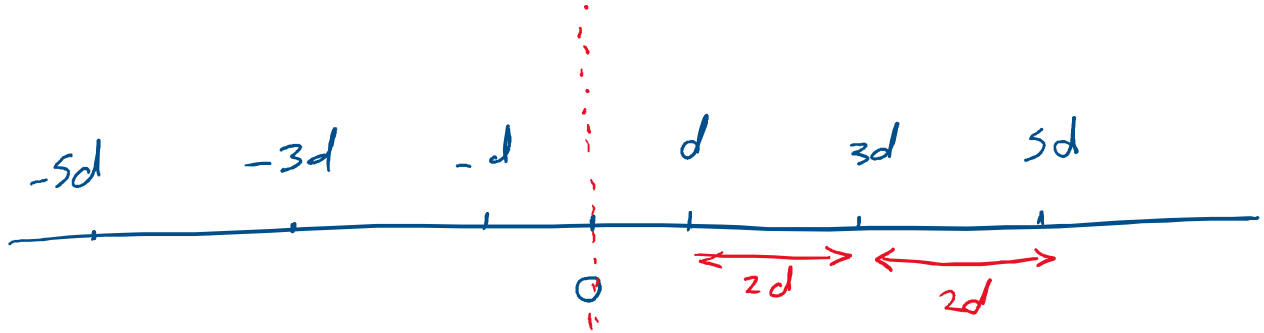
$$\boxed{s_m = \sqrt{\frac{\mathcal{E}_g}{2}}\, A_m}$$

In digital Modulation techinques, it is desirable to have a symmetric signal Constellation. Because this symmetry can help us to design the modulator & demodulator in a simple manner. So, in M-ASK modulation $\{A_m\}_{m=1}^{M}$ are selected such as the signal space symmetry is satisfied. Therefor, $\{A_m\}_{m=1}^{M}$ are usually selected as below;

$$A_m = (2m-1-M)\,d \quad , \quad 1 \leq m \leq M$$

$$\Longrightarrow \quad A_m \in \{\pm d, \pm 3d, \pm 5d, \pm 7d, \ldots, \pm (M-1)d\}$$

modulation levels



The distance between any two adjucent modulatio level is equal to 2d.

Symmetric around zero

$$\Longrightarrow \quad \mathcal{E}_m = \frac{\mathcal{E}_g}{2} A_m^2 = \frac{\mathcal{E}_g}{2} (2m-1-M)^2 d^2$$

second parameter is the average energy of signals $\{s_m(t)\}_{m=1}^{M}$

The average energy of signals $\{s_m(t)\}_{m=1}^{M}$ in M-ASK mod.

is as follows

$$\underbrace{\mathcal{E}_{ave}}_{\text{The average energy of signals}} = \frac{1}{M}\sum_{m=1}^{M} \mathcal{E}_m = \frac{1}{M}\sum_{m=1}^{M} \frac{\mathcal{E}_g}{2} A_m^2 = \frac{\mathcal{E}_g}{2M}\sum_{m=1}^{M} A_m^2$$

$$\mathcal{E}_m = \frac{\mathcal{E}_g}{2} A_m^2$$

$A_m = (2m-1-M)d \Longrightarrow$

$$\mathcal{E}_{ave} = \frac{\mathcal{E}_g}{2M}\sum_{m=1}^{M}(2m-1-M)^2 d^2 = \frac{\mathcal{E}_g d^2}{2M}\sum_{m=1}^{M}(2m-1-M)^2$$

As we know $(2m-1-M)\Big|_{m=1}^{M} = \{\pm d, \pm 3d, \pm 5d, \ldots, \pm (M-1)d\}$

$$\Rightarrow \mathcal{E}_{ave} = \frac{\mathcal{E}_g d^2}{2M} \underbrace{\sum_{m=1}^{M} (2m-1-M)^2}_{} = \frac{\mathcal{E}_g d^2}{\cancel{2M}} \times \cancel{2} \times \frac{1}{6} \cancel{M} (M-1)(M+1)$$

$\underbrace{2 \times \frac{1}{6} M(M-1)(M+1)}_{} \leftarrow$ { Summation of square of odd numbers between 1 to M-1 with + and - signs.

$$\Longrightarrow E_{ave} = \frac{E_g d^2 (M^2 - 1)}{6}$$: The average Energy of Signals in M-ASK Modulation

Third Parameter is the averge Enery Per bit

we know that in M-ary modulation each symbol contain $k = \log_2^M$ bits. So, if we want to compare different M-ary modulations for different values of M, it is not fair to Compare them based on ave. symbol energy. It is fair when we Compare them based on ave. energy per bit. So, we should Calculate the ave. Energy per bit for M-ary mod. techniques

$$E_{b_{ave}} = \frac{E_{ave}}{K} = \frac{E_{ave}}{\log_2 M} = \frac{E_g d^2 (M^2-1)}{6 \log_2 M}$$

average Energy per bit for M-ASK

4^{th} Parameter is Euclidean distance betwee any two signals

The Euclidean distance between any two signal $s_m(t)$ and $s_n(t)$ in the M-ASK signal space is calculated as below;

vector space

$$d_{mn}^{(e)} = \| s_m(t) - s_n(t) \| = \| \underline{s}_m - \underline{s}_n \|$$

$$d_{mn}^{(e)} = \| s_m(t) - s_n(t) \| = \left[\int_0^T \left(s_m(t) - s_n(t) \right)^2 dt \right]^{1/2}$$

$$= \| \underline{s}_m - \underline{s}_n \| = | s_m - s_n | = \left| \sqrt{\frac{\mathcal{E}_g}{2}} A_m - \sqrt{\frac{\mathcal{E}_g}{2}} A_n \right|$$

vector space

$$s_m(t) = A_m \, g(t) \cos 2\pi f_c t = s_m f_1(t), \quad s_m = \sqrt{\frac{\mathcal{E}_g}{2}} A_m$$

$$A_m = (2m-1-M)d$$

$$s_n(t) = A_n \, g(t) \cos 2\pi f_c t = s_n f_1(t), \quad s_n = \sqrt{\frac{\mathcal{E}_g}{2}} A_n$$

$$A_n = (2n-1-M)d$$

$$\Longrightarrow d_{mn}^{(e)} = \sqrt{\frac{\mathcal{E}_g}{2}}\,\left| \underset{\uparrow}{A_m} - \underset{\nwarrow}{A_n} \right| = \sqrt{\frac{\mathcal{E}_g}{2}}\,\left| 2(n-m)d \right| = \sqrt{2\mathcal{E}_g}\,|n-m|d$$

$(2m-1-M)d$ $\qquad$ $(2n-1-M)d$

$$\Longrightarrow d_{mn}^{(e)} = \sqrt{2\mathcal{E}_g}\,|n-m|\,d$$

: Euclidean distance between $s_m(t)$ and $s_n(t)$

5^{th} Parameter is minimum Eculidean distance between Signals

As you can conclude from signal rep. in vector space, the minimum Euclidean distance between signals in M-ASK modulation is where the signals $s_m(t)$ & $s_n(t)$ are adjucent signals. Means $|m-n| = 1$

$$\Longrightarrow \quad d_{min} = \sqrt{2\mathcal{E}_g}\, d = d\sqrt{2\mathcal{E}_g}$$

for M-ASK Signalling

Minimum Euclidean distance between signals

We will see that d_{min} is one of the main factors that affect the modulation performance.

6^{th} parameter is the relation between d_{min} and $\mathcal{E}_{b_{ave}}$

To fairly comparison of different signallings with different values of M, we should do the comparison based $\mathcal{E}_{b_{ave}}$ (average energy per bit). on the other hand, d_{min} is one of the main factors that affect the performance of signallings.

So, relation between d_{min} and $\mathcal{E}_{b_{ave}}$ is important for us to know about each signalling. For M-ASK signalling we can obtain this relation as follows.

$(d_{min}^2)/2$

$$d_{min} = d\sqrt{2\mathcal{E}_g} \quad (1) \quad \text{and} \quad \mathcal{E}_{b_{av}} = \frac{\mathcal{E}_g d^2 (M^2-1)}{6\log_2^M} \quad (2)$$

I want to find the relation of d_{min} based on $\mathcal{E}_{b_{av}}$

1, 2 $\Longrightarrow$ $$\varepsilon_{b_{ave}} = \frac{d_{min}^2 (M^2-1)}{2 \times 6 \log_2 M}$$

$$\Longrightarrow \boxed{d_{min} = \sqrt{\frac{12 \log_2 M}{M^2-1} \varepsilon_{b_{ave}}}}$$

In M-ASK modulation it is important to note to these comments.

* As you know in some Comm. systems, we use PAM signalling in the baseband form, means

$$s_m(t) = A_m \overbrace{p(t)}^{\text{Pulse-shaping signal}} \qquad 1 \leq m \leq M \quad \text{and} \quad 0 \leq t \leq T$$

means that the information is transmitted in the baseband form without any carrier waveform.

Obtain the parameters of baseband as an Excersise.

[Singnal Energies, $\mathcal{E}_{av}$, $\mathcal{E}_{bav}$, $d_{mn}^{(e)}$, d_{min}, d_{min} and $\mathcal{E}_{bave}$ relation]

The other Comment is on the $\overbrace{\text{assignment}}^{\text{mapping}}$ of bits to symbols at the M-ASK modulator input.

bit stream

symbols $\left\{\pm d, \pm 3d, \dots, \pm (M-1)d\right\}\sqrt{\frac{\mathcal{E}_g}{2}}$

$\underbrace{110}10..0$

↑ we have k bit symbols

$$s_m = \sqrt{\frac{\mathcal{E}_g}{2}}\;\underbrace{(2m-1-M)\,d}$$

In M-ASK modulation technique, we assign each k-bits of input bit stream to M-ASK symbols in two different way. The first way is called Natural Coding and the second way is called Gray Coding.

In natural Coding we assign the modulation levels to each k-bits of input stream, in descending (or asending) order. But in Gray Coding we do assignment such that the bit error probability in detection will be minimized.